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q-state Potts model on the checkerboard lattice

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Abstract. A large q expansion of the partition function of the checkerboard q-state Potts model is given up to sixth order in $q^{-1/2}$. This expansion is used to discuss the following two points. First, on the critical manifold, the large q expansion agrees up to this order, with the 'minimal' solution suggested by the group structure associated with this model. Secondly, the latent heat and some correlation functions are discussed from the point of view of their functional dependence in the parameters.

1. Introduction

In a preceding paper (Jaekel and Maillard 1982), the large q expansion was used to check the expression of the partition function of the anisotropic Potts model on the critical curve, and also the expression of the latent heat. Some attention was focused on the independence of the latter in the anisotropy.

In the present paper, we extend this work to the case of the checkerboard lattice, involving four parameters K_1, K_2, K_3 and K_4 (figure 1) instead of two. The partition function of this model verifies an inverse relation (Maillard and Rammal 1983), one can combine to the 'geometrical' symmetries (group of the square C_{4v}) of the model. All these symmetries generate a symmetry group, allowing one to write functional equations satisfied by the partition function. From these functional equations, a 'minimal' solution, on the critical manifold, can be written in the form of an infinite product. In order to verify the validity of this solution, the large q expansion, up to sixth order in $q^{-1/2}$, is used. The agreement obtained is suggestive, considering the large number of parameters of this model.

The presence of many parameters, allows us to look at two interesting questions. The first one is related to the particular dependence of the latent heat L, on these four parameters. In fact, this Potts model can be mapped into an inhomogeneous six-vertex model (see § 3.4). The latter was studied in detail by Baxter (1971), suggesting a very particular expression for the polarisation P. The deduced expression for the latent heat will show a non-trivial dependence on the parameters $\{K_i\}$: the large q expansion confirms this property.

The second question concerns the parameter dependence of a 'diagonal' correlation function, on the anisotropic Potts model ($K_1 = K_3$, $K_2 = K_4$), invariant by the symmetry $K_1 \leftrightarrow K_2$ and also under an automorphy group G (Jaekel and Maillard 1983). In the

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Ising limit ($q = 2$), this correlation function, invariant under the group G, is only a function of the algebraic invariant of this group, $k = \sinh 2K_1 \sinh 2K_2$. We see, using the large q expansion, that this property does not generalise to the Potts model: the diagonal correlation function is not only a function of the algebraic invariant of the group.

2. Diagrams and expansion

Let us denote $a = e^{K_1}$, $b = e^{K_2}$, $c = e^{K_3}$, $d = e^{K_4}$. The partition function per site Z, is given as usual by

$$Z^N(a, b, c, d) = \sum_{\{\sigma\}} \prod_{\langle ij\rangle} a^{\delta_{\sigma_i,\sigma_j}} \prod_{\langle jk\rangle} b^{\delta_{\sigma_j,\sigma_k}} \sum_{\langle kl\rangle} c^{\delta_{\sigma_k,\sigma_l}} \sum_{\langle li\rangle} d^{\delta_{\sigma_l,\sigma_i}} \tag{2.1}$$

where each spin σ belongs to $\mathbb{Z}_q$ and the ordered sequences $\langle ij\rangle$, $\langle jk\rangle$, $\langle kl\rangle$, $\langle li\rangle$ denote the edges of the N plaquettes.

The large q expansion will give the expansion of $\Lambda(a, b, c, d)$ defined by†

$$Z(a, b, c, d) \equiv (1/q)[(a+q-1)(b+q-1)(c+q-1)(d+q-1)]^{1/2}\Lambda(a, b, c, d) \tag{2.2}$$

The parameters of the large q expansion ($T > T_c$) are $1/a^*$, $1/b^*$, $1/c^*$ and $1/d^*$ where $1/a^* = (a-1)/(a+q-1) \to 0$ (similar expressions for $1/b^*$, $1/c^*$ and $1/d^*$).

$$\begin{aligned} \ln \Lambda(a, b, c, d) \\ &= \square (q-1)\frac{1}{a^*}\frac{1}{b^*}\frac{1}{c^*}\frac{1}{d^*} && \Big\} O = 2 \\ &+ \square\!\square\, \tfrac{1}{2}(q-1)(q-2)\frac{1}{a^{*2}c^{*2}}\left(\frac{1}{b^*d^{*2}} + \frac{1}{b^{*2}d^*}\right) && \\ &+ \begin{smallmatrix}\square\\ \square\end{smallmatrix}\, \tfrac{1}{2}(q-1)(q-2)\frac{1}{b^{*2}d^{*2}}\left(\frac{1}{a^*c^{*2}} + \frac{1}{a^{*2}c^*}\right) && \Big\} O = 3 \\ &+ \ldots . \end{aligned} \tag{2.3}$$

In this expansion, one will also use the following expansion parameters

$$\begin{aligned} \frac{1}{a^*} &\equiv t\frac{u-t}{1-ut^3} = o(q^{-1/2}) & \frac{1}{b^*} &\equiv t\frac{v-t}{1-vt^3} = o(q^{-1/2}) \\ \frac{1}{c^*} &\equiv t\frac{w-t}{1-wt^3} = o(q^{-1/2}) & \frac{1}{d} &\equiv t\frac{z-t}{1-zt^3} = o(q^{-1/2}) \end{aligned} \tag{2.4}$$

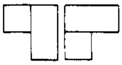
where $t + (1/t) \equiv q^{1/2}$.

The order of each diagram is given by: $O = L - 2B$, where O = order in $q^{-1/2}$ (or t), L = number of links and B = number of loops.

The diagrammatic rules were given in a preceding work (Jaekel and Maillard 1982), and will not be recorded here. Up to 6th order, we have 62 connected and 9 disconnected diagrams. The highest power in q (numbers of loops) is 9, and the largest number of links is 24. The expression of $\ln \Lambda$ calls for two remarks. First, the

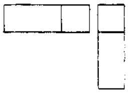
† Note a misprint in equation (2.2) of Jaekel and Maillard (1982a). Λ should be replaced by Λ/q.

invariance with respect to the square group C_{4v}, satisfied by $\ln \Lambda$, has to be taken carefully into account, as can be seen in the following two examples.



$$(q-1)(q-2)\left(\frac{1}{a^*c^{*3}b^{*2}d^{*3}}+\frac{1}{c^*a^{*3}d^{*2}b^{*3}}+\frac{1}{b^*d^{*3}a^{*2}c^{*3}}+\frac{1}{d^*b^{*3}c^{*2}a^{*3}}+\frac{1}{a^{*3}c^*b^{*2}d^{*3}}+\frac{1}{a^*c^{*3}d^{*2}b^{*3}}+\frac{1}{b^{*3}d^*a^{*2}c^{*3}}+\frac{1}{b^*d^{*3}c^{*2}a^{*3}}\right)$$

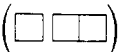
containing 8 terms in opposition to



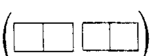
$$(q-1)(q-2)\left[\frac{1}{a^{*3}c^{*3}}\left(\frac{1}{b^*d^{*2}}+\frac{1}{b^{*2}d^*}\right)+\frac{1}{b^{*3}d^{*3}}\left(\frac{1}{a^*c^{*2}}+\frac{1}{a^{*2}c^*}\right)\right]$$

which involves only 4 terms.

Secondly, as usual, the main difficulty lies in the counting of the disconnected diagrams, as seen in the following three examples.

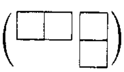


$$(q-1)^2(q-2)(-4)\frac{1}{a^*b^*c^*d^*}\left[\frac{1}{a^{*2}c^{*2}}\left(\frac{1}{b^*d^{*2}}+\frac{1}{b^{*2}d^*}\right)+\frac{1}{b^{*2}d^{*2}}\left(\frac{1}{a^*c^{*2}}+\frac{1}{a^{*2}c^*}\right)\right]$$



$$(q-1)^2(q-2)^2(-\tfrac{7}{4})\frac{1}{a^{*4}b^{*4}c^{*4}d^{*4}}(a^{*2}+b^{*2}+c^{*2}+d^{*2})+(q-1)^2(q-2)^2(-2)\frac{1}{a^{*4}b^{*4}c^{*4}d^{*4}}(a^*c^*+b^*d^*)$$

finally



$$(q-1)^2(q-2)^2(-3)\frac{1}{a^{*4}b^{*4}c^{*4}d^{*4}}(c^*d^*+b^*c^*+a^*d^*+a^*b^*).$$

This development has been computed up to the sixth order in $q^{-1/2}$ (resp. t) and, to facilitate further comparison with other results, $\ln \Lambda$ has been obtained in terms of variables u, v, w, z and t. In order to proceed further in the large q expansion, a more algebraic series expansion technique is needed. For our discussion (see below) the above $o(t^6)$ expansion is sufficient. We write the result in a condensed form,

$$\ln \Lambda = \sum_{i=0}^{6} C_i t^i + o(t^7) \tag{2.5}$$

where

$$C_0 = 0, \qquad C_1 = 0, \qquad C_2 = S_0,$$

$$C_3 = \tfrac{1}{2}S_0 - S_3,$$

$$C_4 = S_0^3 + S_0^2 S_2 - \tfrac{5}{2}S_0^2 - 2S_0S_2 + S_0 + S_2,$$

$$C_5 = -S_1 + \tfrac{1}{2}S_0^4S_1 + 7S_0S_1 + 3S_0^3S_1 - 9S_0^2S_1 + 2S_0^3S_3 - S_3 - 6S_0^2S_3 + \tfrac{11}{2}S_0S_3 + \tfrac{1}{2}S_5 - S_0S_5 + \tfrac{1}{2}S_0^2S_5,$$

$$
\begin{aligned}
C_6 = 1 + S_2 - 3S_4' - 19S_0 - \tfrac{5}{2}S_4 - 12S_0S_2 + S_0^6 \\
&+ S_0^5S_2 + 10S_0^5 + S_0^4S_4' + 6S_0^4S_2 + S_0^3S_6 + S_0^3S_4 - 7S_0^4 \\
&+ 2S_0^3S_4' - 25S_0^3S_2 - \tfrac{15}{4}S_0^2S_6 + 29S_0^2S_2 + 5S_0S_6 - 2S_0^2S_4 \\
&- 10S_0^2S_4' - \tfrac{113}{3}S_0^3 + S_0S_4 + 52S_0^2 + 10S_0S_4'.
\end{aligned}
$$

In the preceding expressions, we have used the following algebraic invariants of the group C_{4v}.

$$
\begin{aligned}
&S_0 = uvwz, \qquad S_1 = u + v + w + z, \\
&S_2 = uv + vw + wz + zu + uw + vz, \qquad S_3 = vwz + uwz + uvz + uvw \\
&S_4 = (vz)^2 + (uz)^2 + (vw)^2 + (uw)^2 + (wz)^2 + (uv)^2, \\
&S_4' = uvz^2 + uwz^2 + u^2wz + u^2vz + uvw^2 + v^2uw + u^2vw + uzv^2 \\
&\qquad + uzw^2 + vwz^2 + wzv^2 + vzw^2, \\
&S_5 = vw^2z^2 + uw^2z^2 + v^2wz^2 + u^2wz^2 + uv^2z^2 + u^2vz^2 + v^2w^2z + u^2w^2z \\
&\qquad + u^2v^2z + uv^2w^2 + u^2vw^2 + u^2v^2w,
\end{aligned}
\tag{2.6}
$$

and

$$S_6 = (uvw)^2 + (uvz)^2 + (uwz)^2 + (vwz)^2.$$

One remarks that, in the special case of the anisotropic Potts model: $u = w$ and $v = z$, one recovers the result obtained for $\ln \Lambda$ in the previously mentioned work.

3. Applications. Partition function, latent heat and correlation functions

The preceding expansion for the partition function can be used to check some conjectures, or natural extensions of well known results, on the checkerboard q-state Potts model. Amongst other things, we will study the partition function, the latent heat and a special correlation function.

3.1. Partition function at criticality

We have already noticed (Maillard and Rammal 1983) that the partition function of the checkerboard q-state Potts model satisfies the inverse relation†

$$Z(u, v, w, z)Z\left(\frac{1}{ut^2}, \frac{t^2}{v}, \frac{1}{wt^2}, \frac{t^2}{z}\right) = \frac{1+t^2}{t^2} \prod_{\alpha = u,w} \left(\frac{(1-\alpha/t)(1-\alpha t^3)}{(1-\alpha t)^2}\right)^{1/2} \tag{3.1}$$

and also the symmetry of the square

$$Z(u, v, w, z) = Z(\tau(u, v, w, z)), \qquad \tau \in C_{4v}. \tag{3.2}$$

These functional equations suggest a 'minimal' solution, on the critical manifold ($uvwz = 1$), for the partition function in the form of an infinite product, produced by the action of the group G generated by the inverse relation (3.1) and the symmetries

† The notion of inverse relation was first introduced in statistical mechanics by Stroganov (1979) and extensively used by Baxter (1980).

(3.2). This solution can be written

$$Z^2(u, v, w, z) = \frac{q}{t^2} \frac{F(u)F(1/u)}{1-tu} \frac{F(v)F(1/v)}{1-tv} \frac{F(w)F(1/w)}{1-tw} \frac{F(z)F(1/z)}{1-tz} \tag{3.3}$$

with

$$uvwz = 1 \tag{3.4}$$

and where

$$F(u) = \prod_{n=1}^{\infty} \frac{1-t^{4n-1}u}{1-t^{4n+1}u}. \tag{3.5}$$

From (3.3), we deduce

$$\Lambda^2(u, v, w, z) = (qt^2)\frac{F(u)F(1/u)}{1-t^3u} \frac{F(v)F(1/v)}{1-t^3v} \frac{F(w)F(1/w)}{1-t^3w} \frac{F(z)F(1/z)}{1-t^3z}. \tag{3.6}$$

One remarks that (3.3) reproduces the known results (Baxter *et al* 1978, Maillard and Rammal 1983) for the anisotropic square lattice ($u = w, v = z$), for the triangular lattice ($z \to 1/t$) and for the honeycomb lattice ($z \to t$).

Expanding (3.6), up to the sixth order in t, one obtains

$$\begin{aligned} \ln \Lambda(u, v, w, z) & \\ &= t^2 - \tfrac{1}{2}t^3(vwz + uwz + uvz + uvw) - \tfrac{1}{2}t^4 \\ &\quad + \tfrac{1}{2}t^5(u + v + w + z + vwz + uvz + uvw + uwz) \\ &\quad + t^6[\tfrac{1}{3} - \tfrac{1}{4}(v^2w^2z^2 + u^2w^2z^2 + u^2v^2z^2 + u^2v^2w^2)] + o(t^7) \end{aligned} \tag{3.7}$$

(3.7) coincides with the large q expansion result (2.5) at $S_0 \equiv uvwz = 1$. This remarkable agreement, despite the great number of variables, supports the validity of (3.3) for the exact expression of the critical partition function at $q > 4$.

Of course a part of the agreement between (3.6) and (3.7) is a direct consequence of their C_{4v} invariance. However, a more detailed study is needed to see how much of the agreement is non-trivial. (Equation (3.7) is a large q expansion of a free energy having precise analytical and thermodynamic properties (convexity . . .) and is invariant under G. On the other hand (3.6) is invariant under G, but its precise expression is obtained finally from the minimality assumption.)

3.2. *The latent heat*

It is known that the latent heat L, of the anisotropic Potts model (for $q > 4$), up to a known factor, has a simple expression which does not depend on the anisotropy of the model (Baxter 1973). It is natural to ask if this is a general property, that is to say if we have

$$L \equiv -\frac{\partial}{\partial\beta} \ln Z|_{T \geq T_c} + \frac{\partial}{\partial\beta} \ln Z|_{T \leq T_c} = \left(\frac{1}{u}\frac{du}{d\beta} + \frac{1}{v}\frac{dv}{d\beta} + \frac{1}{w}\frac{dw}{d\beta} + \frac{1}{z}\frac{dz}{d\beta}\right)\Bigg|_{uvwz=1} P \tag{3.8}$$

where P is only a function of t, given by

$$P = \prod_{n=1}^{\infty} \frac{1-t^{2n}}{1+t^{2n}} \tag{3.9}$$

(as usual β denotes $1/T$).

Using the large q expansion (2.5), we can answer this question positively. Considering the high-temperature and low-temperature versions of the large q expansion (via the duality relations), and taking the difference of the corresponding first derivatives, one gets from (2.5)

$$L=\frac{\partial}{\partial\beta}\left(\ln\Lambda\left(\frac{1}{u},\frac{1}{v},\frac{1}{w},\frac{1}{z}\right)-\ln\Lambda(u,v,w,z)\right.$$
$$\left.+\frac{1}{2}\frac{u(1-t^3/u)}{1-t^3u}\frac{v(1-t^3/v)}{1-t^3v}\frac{w(1-t^3/w)}{1-t^3w}\frac{z(1-t^3/z)}{1-t^3z}\right)_{uvwz=1}$$
$$=\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}\beta}+\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\beta}+\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}\beta}+\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}\beta}\right)\bigg|_{uvwz=1}(1-4t^2+4t^4+\mathrm{o}(t^7)). \tag{3.10}$$

Hence, this remarkably simple dependence of L, on the variables u, v, w and z, is supported by our large q expansion†.

3.3 Next-nearest neighbour two-point correlation function

Recently, the inverse relation ideas have been generalised to obtain functional relations satisfied by a spin–spin correlation function (Jaekel and Maillard 1983). For instance, in the case of the simple anisotropic Potts model, the correlation function between next-nearest-neighbour spins (see figure 1), satisfies the inverse relations

$$\langle\delta_{\sigma_i,\sigma_k}\rangle(a,b)=\langle\delta_{\sigma_i,\sigma_k}\rangle(1/a,2-q-b) \tag{3.11}$$

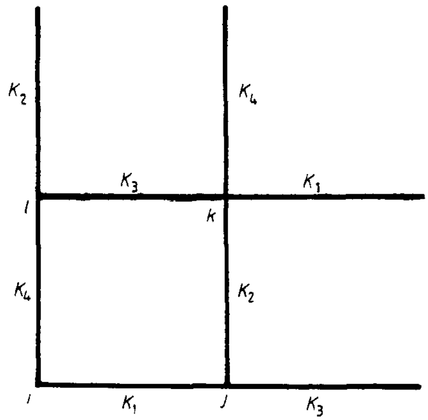


Figure 1. Elementary cell of the checkerboard lattice.

for every a and b. This correlation function is obviously symmetric in a and b. By consequence, $\langle\delta_{\sigma_i,\sigma_k}\rangle$ is invariant under the action of the group generated by these two transformations.

In the Ising limit ($q=2$), $\langle\delta_{\sigma_i,\sigma_k}\rangle$ reduces to a function of the algebraic invariant ($k=\sinh 2K_1\sinh 2K_2$) of the symmetry group. In the anisotropic Potts model, the corresponding algebraic invariant is nothing other than the product uv. Hence, it is legitimate to check the dependence of $\langle\delta_{\sigma_i,\sigma_k}\rangle$, in the variable uv. The expansion (2.5)

† Note a misprint in the first line of equation (3.5) of Jaekel and Maillard (1982). t should be replaced by t^3. The second line remains unchanged.

allows us to answer this question. In fact $\langle \delta_{\sigma_i,\sigma_k} \rangle$ can be obtained by carrying out the two following operations.

(i) Taking the limit $K_4 \to \infty$ in the expression of $\ln Z$, one obtains the triangular lattice limit.

(ii) taking the derivative of the previous expression with respect to K_3, at $K_3 = 0$, one gets $\langle \delta_{\sigma_i,\sigma_k} \rangle$ up to inessential factor.

Thus, to obtain the dependence of $\langle \delta_{\sigma_i,\sigma_k} \rangle$ in the uv variable, it is sufficient to calculate $(\partial/\partial w) \ln \Lambda\,(u, v, w, 1/t)$ at $w = t$.

From the expression (2.5), one can see that $\langle \delta_{\sigma_i,\sigma_k} \rangle$, which is invariant under the symmetry group, does not depend only on the simple algebraic invariant uv.

4. Correspondence with generalised six-vertex model

The 'equivalence' of the Potts model in two dimensions, with an ice-rule vertex model is well known (see Wu for a review, 1982). In the case of the checkerboard lattice, the corresponding six-vertex model is a spatial case of the generalised ferroelectric model on a square lattice, studied by Baxter (1971). This correspondence is detailed explicitly in the appendix. The important point to be noted here is the equivalence between the integrability condition of the generalised ferroelectric model (equation (1.7) in Baxter 1971) and the criticality condition of the checkerboard model. It seems that this equivalence between these two conditions is general. In particular, we verify indirectly the exactness of the expression (3.3) for the partition function suggested by our group considerations (equation (1.8) in Baxter, 1971). The same argument holds also for the expression of the latent heat L, given by equation (3.8), where P corresponds exactly to the polarisation of the vertex model. In this way, the particular dependence of the latent heat on the model parameters, is the counterpart of that mentioned by Baxter for the polarisation.

5. Conclusion

In this paper, a large q expansion of the checkerboard q-state Potts model was given up to the sixth order in $q^{-1/2}$. This result was used to discuss essentially three points. First, we have checked the agreement of this expansion with the minimal solution for the partition function at criticality. Secondly, we have also checked the expression of the latent heat. Finally, the dependence of the next-nearest-neighbour two-point correlation function, on the algebraic invariant of the automorphy group, was discussed for the anisotropic q-state Potts model and we have seen that the correlation function despite its G invariance is not a function of the only algebraic invariant of G in contrast to the Ising limit. Having several parameters, u, v, w, z and t at our disposal, the large q expansion can be used to infirm or confirm many statements on different physical quantities (by making elementary operations as derivation, limits, . . .). In another way, using this approach, we gain more insight into the complexity of the exact solution, away from criticality.

This simple approach, which consists of exhibiting the 'minimal' solution associated with the automorphy group, and then to check its validity by diagrammatic calculation, can be seen in opposition to standard methods (Bethe-ansatz, star–triangle relations . . .). Despite its simplicity, this method, in counterpart, does not at present give very precise information such as the eigenvectors or correlation functions.

Acknowledgments

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Appendix 1

The Potts model, in two dimensions, is equivalent to an ice-rule vertex model. This equivalence, first pointed out by Temperley and Lieb (1971) for the square lattice, has been generalised by Baxter *et al* (1976), to different lattices. Such correspondence can be extended also for the checkerboard lattice. We state only the result. For this lattice, we have four sublattices on the corresponding vertex model, denoted 1, 2, 3 and 4 (figure 2). The weights are given by

$$(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = (1, 1, x_i, x_i, A_i, B_i)$$

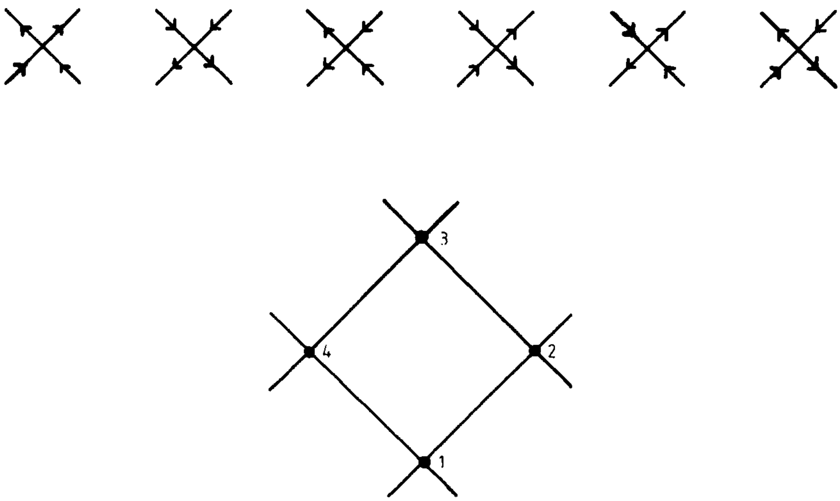

Figure 2. The standard six vertex, of respective weights $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$ for sublattices 1 and 3; $(\omega_3, \omega_4, \omega_2, \omega_1, \omega_6, \omega_5)$ for sublattices 2 and 4.

where

$$i = 1, 2, 3, 4; \qquad x_i = q^{-1/2}(e^{K_i} - 1)$$
$$A_i = t^{-1/2} + x_i t^{1/2}, \qquad B_i = t^{1/2} + x_i t^{-1/2}.$$

The following conventions for the vertex weights on different sublattices are to be used

$$(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) \text{ for sublattices 1 and 3}$$

and

$$(\omega_3, \omega_4, \omega_2, \omega_1, \omega_6, \omega_5) \text{ for sublattices 2 and 4.}$$

This vertex model can be viewed as a special case of the generalised ferroelectric model on a square lattice (Baxter 1971). Using Baxter notations, we are in the case

$\alpha(I, J) = \beta(I, J) = \gamma(I, J) = 1$, and

$$\chi(I, J) = (1 - tp(I, J))^{-1}$$

the parameter t has the same meaning, and only four parameters $p(I, J)$ are to be used. If we denote these parameters by p_1, p_2, p_3 and p_4 (p_i for sublattices $i, i = 1, 2, 3, 4$), one obtains, after some simple calculations, $p_1 = u, p_2 = 1/v, p_3 = w$ and $p_4 = 1/z$. Thus the integrability condition $p(I, J) = \rho(I)\sigma(J)$, which reduces to $p_1p_3 = p_2p_4$ is equivalent to the criticality condition $uvwz = 1$.

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